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Fork-decomposition of strong product of graphs

A. Samuel Issacraj*
J. Paulraj Joseph†

Abstract

Decomposition of arbitrary graphs into subgraphs of small size is assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths, cycles, trees, stars, sunlet etc. Fork is a tree obtained by subdividing any edge of a star of size three exactly once. In this paper, we investigate the necessary and sufficient for the fork-decomposition of Strong product of graphs.
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*Research Scholar (Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, INDIA); samuelissacraj@gmail.com
†Professor (Department of Mathematics, Manonmaniam Sundaranar University Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, INDIA); prof.jpaulraj@gmail.com.
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1 Introduction

We consider only simple, finite and undirected graphs. Let K_n denote the complete graph on n vertices and $K_{m,n}$ denote the complete bipartite graph with parts of sizes m and n. Let P_k denote the path of length $k-1$ and S_k denote the star of size $k-1$. Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An L-decomposition of G is an edge-disjoint decomposition of G into positive integers α_i copies of H_i where $i \in \{1, 2, ..., r\}$. Furthermore, if each $H_i (i \in \{1, 2, ..., r\})$ is isomorphic to a graph H, then we say that G has an H-decomposition. The obvious necessary condition for the existence of a $\{H_1, H_2, ..., H_r\}$ - decomposition of G is

$$\sum_{i=1}^{r} \alpha_i e(H_i) = e(G) \tag{1}$$

We call this equation as necessary sum condition.

Terms not defined here are used in the sense of Bondy and Murty Bondy and Murty [2008].

Decomposition of arbitrary graphs into subgraphs of small size are assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths qing Zhai and hong Lu [2006], Kumar [2003], cycles Alspach and Gavlas [2001], trees Barat and Gerbner [2014], stars Zhao and Wu [2015], sunlet Akwu and Ajayi [2013] etc. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi Dor and Tarsi [1997].

Fork is a tree obtained by subdividing any edge of a star of size three exactly once. A tree with degree sequence $(1, 1, 1, 2, 3)$ is unique and is nothing but the fork defined above. This graph was defined by Simone and Sassano in the name of *chair graph* in 1993, when they studied the stability number of bull and chair-free graphs Simone and Sassano [1993]. In 2014, Barat and Gerbner Barat and Gerbner [2014] studied decomposition of 191-edge connected graphs which can be decomposed into forks as a possible attempt to solve the following conjecture:

Conjecture 1. Barat and Thomassen [2006] For each tree T, there exists a natural number k_T such that the following holds: if G is a k_T -edge-connected simple graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.

The edge-connectivity constants in the solved cases of Conjecture 1 are seemingly far from best possible. There is very little known about lower bounds. This motivated us to concentrate on strong product graphs which have lower edge-connectivity.

If a graph G admits a H-decomposition, then $|E(H)|$ divides $|E(G)|$. Since the size of a fork is 4, for a fork-decomposition the obvious necessary condition is

$$|E(G)| \equiv 0 \ (mod\ 4) \tag{2}$$

In Devi and Joseph [2013] they studied the P_4 Decomposition of Product graphs. Decomposition of complete bipartite graphs, complete graphs and corona graphs into Fork was studied in Joseph and Issacraj [2022]. Fork-decomposition of cartesian product and direct product of graphs were studied in Issacraj and Joseph [2023] & Issacraj and Joseph [2022] respectively. In this paper, we investigate the decomposition of strong product of graphs into forks.

Definition 1.1. *Hammack et al. [2011] The strong product of G and H is the graph denoted as $G \boxtimes H$, and defined by $V(G \boxtimes H) = \{(g,h)/g \in V(G)$ and $h \in V(H)\}$, $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$. The number of edges in $G \boxtimes H$ is $|V(G)||E(H)| + |V(H)||E(G)| + 2|E(G)||E(H)|$.*

Notation 1.1. *We observe that $K_2 \boxtimes K_2 = K_4$. The product of K_2 with itself produces the shape $\boxtimes$. This explains the reason behind the symbol for this product.*

The following results are used in the subsequent section.

Theorem 1.1. *Joseph and Issacraj [2022] The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \ (mod\ 4)$ except $K_{2,4i+2}(i = 1, 2, ...)$.*

Theorem 1.2. *Joseph and Issacraj [2022] $C_n \circ \overline{K_m}$ is fork-decomposable if and only if $m = 1$ and $n = 2k$ or $m = 3$.*

Theorem 1.3. *Issacraj and Joseph [2023] $P_n \square P_m$ is fork-decomposable if and only if $m \equiv n \ (mod\ 4)$ where $n \leq m$.*

Theorem 1.4. *Issacraj and Joseph [2023] The graph $K_m \square P_n$ is fork-decomposable if it satisfies any one of the following conditions.*

1. $m \equiv 0 \ (mod\ 8)$
2. $n \equiv 0 \ (mod\ 2)$ *and* $m \equiv 0 \ (mod\ 4)$
3. $n \equiv 2 \ (mod\ 4)$ *and* $m \equiv 2 \ (mod\ 4)$
4. $n \equiv 3 \ (mod\ 4)$ *and* $m \equiv 5 \ (mod\ 8)$
5. $n \equiv 1 \ (mod\ 4)$ *and* $m \equiv 1 \ (mod\ 8)$

Theorem 1.5. *Issacraj and Joseph [2023] The graph $K_m \square C_n$ is fork-decomposable if and only if it satisfies any one of the following conditions.*

1. n *is even and* $m \equiv 0 \ (mod\ 4)$ *or* $m \equiv -1 \ (mod\ 4)$

2. *n is odd and $m \equiv 0 \ (mod\, 8)$ or $m \equiv -1 \ (mod\, 8)$*

3. *$n \equiv 0 \ (mod\, 4)$*

Theorem 1.6. *Issacraj and Joseph [2022] For $m, n \geq 3$, $P_m \times P_n$ is fork-decomposable if and only if $(m-1)(n-1)$ is a multiple of 4.*

Theorem 1.7. *Issacraj and Joseph [2022] For $m \geq 3$, $K_m \times P_n$ is fork-decomposable if and only if*

1. *n is even and $m = 4k$ or $4k+1$.*

2. *n is odd.*

Theorem 1.8. *Issacraj and Joseph [2022] For $m \geq 3$, $K_m \times C_n$ is fork-decomposable if and only if*

1. *n is odd and $m = 4k$ or $4k+1$.*

2. *n is even.*

In this paper, we investigate the necessary and sufficient condition for the decomposition of strong product of graphs into forks.

2 Strong product of two paths

In this section, we investigate the necessary and sufficient condition for the decomposition of strong product of paths into forks. First let us see an example for the fork-decomposition of strong product of paths.

Example 2.1. *$P_2 \boxtimes P_4$ is fork-decomposable.*

Proof. Let $V(P_2) = \{x_1, x_2\}$ and $V(P_4) = \{y_1, y_2, y_3, y_4\}$. Then $V(P_2 \boxtimes P_4) = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)\}$. Rename the vertices $(x_1, y_i) = u_i$, and $(x_2, y_i) = v_i$, for all $1 \leq i \leq 4$. Fork-decomposition of $P_2 \boxtimes P_4$ is given in Figure 1.

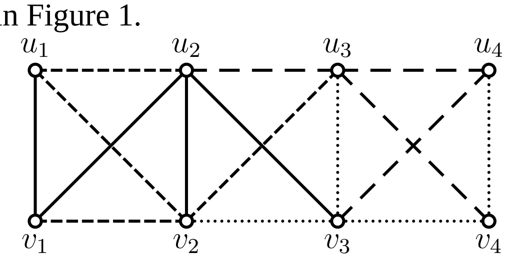

Figure 1: $P_2 \boxtimes P_4$ □

The following two lemmas are needed for proving the theorem.

Lemma 2.1. *For* $m \geq 2$, $P_2 \boxtimes P_m$ *is fork-decomposable if and only if* $m \equiv 0 \ (mod\ 4)$.

Proof. Total number of edges in $P_2 \boxtimes P_m$ is $5m - 4$. If $P_2 \boxtimes P_m$ is fork - decomposable, then $5m - 4 \equiv 0 \ (mod\ 4)$. This implies that $m \equiv 0 \ (mod\ 4)$.

Conversely, assume that $m \equiv 0 (mod\ 4)$.

Let $V(P_2) = \{x_1, x_2\}$ and $V(P_m) = \{y_1, y_2, \ldots, y_m\}$. Then $V(P_2 \boxtimes P_m) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m)\}$. Rename the vertices $(x_1, y_i) = u_i$, and $(x_2, y_i) = v_i$, for all $1 \leq i \leq m$.

If $m = 4$, $P_2 \boxtimes P_4$ is fork-decomposable by Example 2.1.

Fork-decomposition of the graph obtained after removing $P_2 \boxtimes P_4$ from $P_2 \boxtimes P_m$ is given by $\{u_{4i+1}v_{4i},\ u_{4i+1}v_{4i+1},\ u_{4i+1}v_{4i+2},\ v_{4i+2}u_{4i+2}\}$, $\{v_{4i+1}u_{4i},\ v_{4i+1}v_{4i},\ v_{4i+1}v_{4i+2},\ u_{4i}u_{4i+1}\}$, $\{u_{4i+2}u_{4i+1},\ u_{4i+2}v_{4i+1},\ u_{4i+2}v_{4i+3},\ v_{4i+3}v_{4i+2}\}$, $\{u_{4i+3}u_{4i+2},\ u_{4i+3}v_{4i+3},\ u_{4i+3}v_{4i+4},\ v_{4i+4}v_{4i+3}\}$, $\{u_{4i+4}v_{4i+4},\ u_{4i+4}v_{4i+3},\ u_{4i+4}u_{4i+3}, u_{4i+3}v_{4i+2}\}$, for $1 \leq i \leq \frac{m-4}{4}$. Hence the graph $P_2 \boxtimes P_m$ is fork-decomposable.

□

Lemma 2.2. *For* $m \geq 2$, $P_4 \boxtimes P_m$ *is fork-decomposable if and only if* $m \equiv 2 \ (mod\ 4)$.

Proof. Let $G = P_4 \boxtimes P_m$. Total number of edges in $P_4 \boxtimes P_m$ is $17m - 10$. If G is fork-decomposable, then $17m - 10 \equiv 0 \ (mod\ 4)$ which implies that $8(2m - 1) + m - 2 \equiv 0 \ (mod\ 4)$. Then, $m - 2 \equiv 0 \ (mod\ 4)$. This implies that $m \equiv 2 \ (mod\ 4)$.

Conversely, assume that $m \equiv 2 \ (mod\ 4)$.

Let $V(P_4) = \{x_1, x_2, x_3, x_4\}$ and $V(P_m) = \{y_1, y_2, \ldots, y_m\}$. Then $V(P_4 \boxtimes P_m) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m), (x_3, y_1), (x_3, y_2), \ldots, (x_3, y_m), (x_4, y_1), (x_4, y_2), \ldots, (x_4, y_m)\}$.
Rename the vertices $(x_1, y_i) = a_i$, $(x_2, y_i) = b_i$, $(x_3, y_i) = c_i$ and $(x_4, y_i) = d_i$, for all $1 \leq i \leq m$.

For $m = 2$. The graph is isomorphic to $P_2 \boxtimes P_4$ which is fork-decomposable by Lemma 2.1. Let H_1 be the graph obtained by removing $P_4 \boxtimes P_2$ from $P_4 \boxtimes P_m$. Then a fork-decomposition of H_1 is given by $\{a_{i+1}a_i, a_{i+1}b_{i+1}, a_{i+1}a_{i+2}, b_{i+1}c_i\}$, $\{b_ia_{i+1}, b_ib_{i+1}, b_ic_{i+1}, b_{i+1}a_i\}$, $\{d_{i+1}d_i, d_{i+1}c_{i+1}, d_{i+1}c_{i+2}, b_{i+1}c_{i+1}\}$, $\{c_{i+1}c_i, c_{i+1}d_i, c_{i+1}c_{i+2}, c_{i+2}b_{i+1}\}$, $\{b_{i+2}a_{i+1}, b_{i+2}b_{i+1}, b_{i+2}c_{i+1}, b_{i+1}a_{i+2}\}$, $\{d_{i+2}c_{i+1}, d_{i+2}d_{i+1}, d_{i+2}c_{i+3}, d_{i+1}c_i\}$, $\{a_{i+2}a_{i+3}, a_{i+2}b_{i+2}, a_{i+2}b_{i+3}, b_{i+2}c_{i+2}\}$, $\{c_{i+2}b_{i+3}, c_{i+2}c_{i+3}, c_{i+2}d_{i+2}, b_{i+3}b_{i+2}\}$, $\{d_{i+3}c_{i+2}, d_{i+3}c_{i+3}, d_{i+3}d_{i+4}, c_{i+3}b_{i+3}\}$, $\{a_{i+3}b_{i+2}, a_{i+3}b_{i+3}, a_{i+3}b_{i+4}, b_{i+2}c_{i+3}\}$, $\{a_{i+4}a_{i+3}, a_{i+4}b_{i+3}, a_{i+4}b_{i+4}, b_{i+3}c_{i+4}\}$, $\{b_{i+4}b_{i+3}, b_{i+4}c_{i+3}, b_{i+4}c_{i+4}, c_{i+3}d_{i+4}\}$, $\{c_{i+4}c_{i+3}, c_{i+4}d_{i+3}, c_{i+4}d_{i+4}, d_{i+3}d_{i+2}\}$, for $i \equiv 2\ (mod\ 4)$. $G = H_1 \cup (P_4 \boxtimes P_2)$. Since H_1 and $P_4 \boxtimes P_2$ are fork-decomposable, G is fork-decomposable. □

In the following theorem, we investigate the necessary and sufficient condition for the fork-decomposition of strong product of paths.

Theorem 2.1. *For $m, n \geq 2$, $P_m \boxtimes P_n$ is fork-decomposable if and only if it satisfies any one of the following conditions:*

1. $m \equiv 0\ (mod\ 4)$ *and* $n \equiv 2\ (mod\ 4)$
2. $m \equiv 1\ (mod\ 4)$ *and* $n \equiv 1\ (mod\ 4)$
3. $m \equiv 3\ (mod\ 4)$ *and* $n \equiv 3\ (mod\ 4)$.

Proof. Total number of edges in $P_m \boxtimes P_n$ is $n(m-1)+m(n-1)+2(m-1)(n-1) = 4mn - 3(m+n) + 2$.

If $P_m \boxtimes P_n$ is fork-decomposable, then $4mn-3(m+n)+2$ must be a multiple of 4. That means $3(m+n)-2$ must be a multiple of 4. Then $3(m+n)-2 = 4p$, where p is any arbitrary positive integer. Then $3(m+n) = 4p+2 = 2(2p+1)$ which implies that $m+n$ must be even. Hence m and n must be both even or both odd. Let the values of m and n be $(4k, 4k+1, 4k+2, 4k+3)$ and $(4l, 4l+1, 4l+2, 4l+3)$ respectively. If m is $4k$ then n can take the values $4l$ or $4l+2$. If m is $4k+1$ then n can take the values $4l+1$ or $4l+3$. If m is $4k+2$ then n can take the values $4l$ or $4l+2$. If m is $4k+3$ then n can take the values $4l+1$ or $4l+3$.

Suppose $m = 4k$ and $n = 4l$, then $3(m+n)-2 = 3(4k+4l)-2 = 12(k+l)-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k$ and $n = 4l+2$, then $3(m+n)-2 = 3(4k+4l+2)-2 = 12(k+l)+6-2$ which is a multiple of 4. Hence $m \equiv 0\ (mod\ 4)$ and $n \equiv 2\ (mod\ 4)$ which is a condition 1.

Suppose $m = 4k+1$ and $n = 4l+1$, then $3(m+n)-2 = 3(4k+1+4l+1)-2 = 12(k+l)+6-2$ which is a multiple of 4. Hence $m \equiv 1\ (mod\ 4)$ and $n \equiv 1\ (mod\ 4)$ which is a condition 2.

Suppose $m = 4k+1$ and $n = 4l+3$, then $3(m+n)-2 = 3(4k+1+4l+3)-2 = 12(k+l)+12-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k+2$ and $n = 4l+2$, then $3(m+n)-2 = 3(4k+2+4l+2)-2 = 12(k+l)+12-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k+3$ and $n = 4l+3$, then $3(m+n)-2 = 3(4k+3+4l+3)-2 = 12(k+l)+18-2$ which is a multiple of 4. Hence $m \equiv 3\ (mod\ 4)$ and $n \equiv 3\ (mod\ 4)$ which is a condition 3.

Now we shall prove the converse part.

Let $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots, y_n\}$.

Then $V(P_m \boxtimes P_n) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m), \ldots, (x_n, y_1), (x_n, y_2), \ldots, (x_n, y_m)\}$. Rename the vertices $(x_i, y_j) = u_{ij}$, for all $1 \le i \le m$ and $1 \le j \le n$.

Suppose $m \equiv 0\ (mod\ 4)$ and $n \equiv 2\ (mod\ 4)$. The induced subgraph $\langle x_{i1}, x_{i2}, \ldots, x_{in}\rangle$ $i = 1, 2$ is isomorphic to $P_2 \boxtimes P_n$ which is fork-decomposable, by Lemma 2.1. The induced subgraph $\langle x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{(i+1)1}, x_{(i+1)2}, x_{(i+1)3}, x_{(i+1)4}, \ldots, x_{(i+4)1}, x_{(i+4)2}, x_{(i+4)3}, x_{(i+4)4}\rangle, -\langle x_{i1}, x_{i2}, x_{i3}, x_{i4}\rangle$, $i \equiv 2\ (mod\ 4)$ is isomorphic to H_1 (given in Lemma 2.2) which is fork-decomposable.

After removing $P_2 \boxtimes P_n$ and H_1 from $P_m \boxtimes P_n$, the graph H_2 given in the figure 2 is obtained.

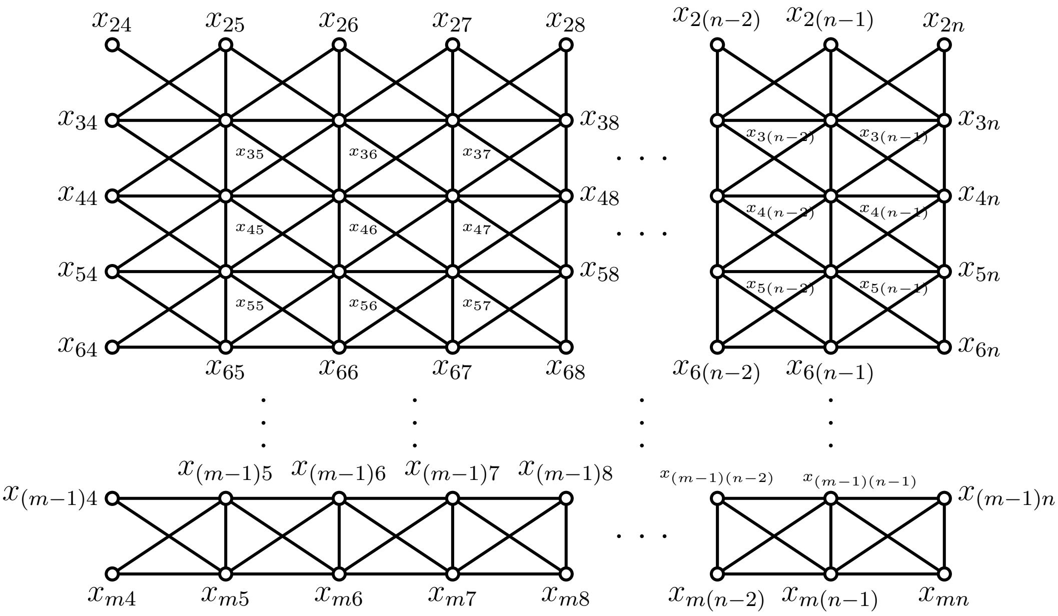

Figure 2: The graph H_2.

The graph H_2 can be decomposed into $P_{m-1} \times P_{n-3}$ and collection of forks $\{x_{i(j+1)}x_{ij},\ x_{i(j+1)}x_{(i-1)(j+1)},\ x_{i(j+1)}x_{i(j+2)},\ x_{i(j+2)}x_{(i-1)(j+2)}\}$ where $i = 3, 4, \ldots, m,\ j \equiv 0\ (mod\ 2),\ 4 \le j \le m-2$. The graph $P_{m-1} \times P_{n-3}$ is fork-decomposable by Theorem 1.6, since $m - 1 \equiv 0\ (mod\ 4) \Rightarrow m \equiv 1\ (mod\ 4)$ and $n - 3 - 2 \equiv 0\ (mod\ 4) \Rightarrow n \equiv 5\ (mod\ 4) \Rightarrow n \equiv 1\ (mod\ 4)$. The graph $P_m \boxtimes P_n = (P_2 \boxtimes P_n) \cup H_1 \cup H_2$ and hence $P_m \boxtimes P_n$ is fork-decomposable.

The graph $P_m \boxtimes P_n = (P_m \square P_n) \cup (P_m \times P_n)$. By Theorem 1.3 and 1.6, $P_m \boxtimes P_n$ is fork-decomposable for conditions (2) and (3). $\square$

3 Strong product of path and cycle

In the following theorem, we investigate the necessary and sufficient condition for the decomposition of strong product of path and cycle into forks.

Theorem 3.1. *For* $m \geq 3$, $C_m \boxtimes P_n$ *is fork-decomposable if and only if* $m \equiv 0\,(mod\,4)$.

Proof. Total number of edges in $C_m \boxtimes P_n$ is $4mn - 3m$.

If $C_m \boxtimes P_n$ is fork-decomposable, by Equation (2) $4mn - 3m \equiv 0\,(mod\,4)$. This implies that $m(4n-3) \equiv 0\,(mod\,4)$. Then, $m \equiv 0\,(mod\,4)$ or $4n-3 \equiv 0\,(mod\,4)$. The second condition does not hold and hence $m \equiv 0\,(mod\,4)$.

Conversely, assume that $m \equiv 0\,(mod\,4)$.

First let us prove that $C_m \boxtimes P_2$ is fork-decomposable.

Let $V(C_n) = \{w_1,\ w_2,\ \ldots, w_n\}$ and $V(P_2) = \{x_1, x_2\}$. Then $V(C_m \boxtimes P_2) = \{(w_1, x_1),\ (w_2, x_1), \ldots, (w_n, x_1),\ (w_1, x_1),\ (w_2, x_2),\ \ldots,\ (w_n, x_2)\}$. Rename the vertices $(w_j, x_1) = u_j$, $(w_k, x_2) = v_k, 1 \leq j, k \leq m$. Then the fork-decomposition of $C_m \boxtimes P_2$ is given by $\{u_iv_i,\ u_iv_{i+1},\ u_iv_{i-1},\ v_{i+1}u_{i+1}\}$, $\{v_iv_{i+1}, v_iu_{i-1}, v_iv_{i-1}, u_{i-1}u_i\}$, $\{u_{i+1}u_i, u_{i+1}v_i, u_{i+1}v_{i+2}, v_{i+2}v_{i+3}\}$, $\{u_{i+2}u_{i+1}, u_{i+2}v_{i+2}, u_{i+2}v_{i+3}, v_{i+2}v_{i+1}\}$, $\{u_{i+3}u_{i+2}, u_{i+3}v_{i+3}, u_{i+3}v_{i+2}, u_{i+2}v_{i+1}\}$ for $i \equiv 1\,(mod\,4)$. The subscripts are taken modulo m.

The graph $C_m \boxtimes P_n$ can be decomposed into one copy of $C_m \boxtimes P_2$ and $n-3$ copies of $C_n \circ \overline{K_3}$. $C_n \circ \overline{K_3}$ is fork-decomposable by Theorem 1.2. Hence $C_m \boxtimes P_n$ is fork-decomposable. □

In the following theorem, we investigate the necessary and sufficient condition for the decomposition of strong product of cycles into forks.

Theorem 3.2. *For* $m, n \geq 3$, $C_m \boxtimes C_n$ *is fork-decomposable for all values of* m *and* n.

Proof. Total number of edges in $C_m \boxtimes C_n$ is $4mn$. If $C_m \boxtimes C_n$ is fork - decomposable, then $|E(C_m \boxtimes C_n)|$ satisfies Equation (2) for all values of m and n.

The graph $C_m \boxtimes C_n$ can be decomposed into n copies of $C_n \circ \overline{K_3}$. $C_n \circ \overline{K_3}$ is fork-decomposable by Theorem 1.2. Hence $C_m \boxtimes C_n$ is fork-decomposable. □

4 Strong product of complete graph and path

In the following theorem, we state the necessary and sufficient condition for the fork-decomposition of strong product of complete graph and path.

Theorem 4.1. *The graph* $K_m \boxtimes P_n$ *if fork-decomposable if and only if satisfies any one of the following conditions:*

1. $m \equiv 0\ (mod\ 8)$

2. $n \equiv 1\ (mod\ 4)$ *and* $m \equiv 1\ (mod\ 8)$

3. $n \equiv 2\ (mod\ 4)$ *and* $m \equiv 0\ (mod\ 4)$

4. $n \equiv 3\ (mod\ 4)$ *and* $m \equiv 5\ (mod\ 8)$

5. $n \equiv 0\ (mod\ 4)$ *and* $m \equiv 0\ (mod\ 2)$

Proof. Total number of edges in $K_m \boxtimes P_n$ is $m(n-1) + n\left(\frac{m(m-1)}{2}\right) + 2(n-1)\left(\frac{m(m-1)}{2}\right) = \frac{3m^2n-2m^2-mn}{2} = \frac{m}{2}\left(m(3n-2)-n\right)$.

If the graph $K_m \boxtimes P_n$ is fork - decomposable, then by Equation (2), $\frac{m}{2}\left(m(3n-2)-n\right) \equiv 0\ (mod\ 4)$. That is,

$$m(m(3n-2)-n) \equiv 0\ (mod\ 8) \tag{3}$$

Clearly, it satisfies $m \equiv 0\ (mod\ 8)$ which is a condition (1).

Assume that $n \equiv 0\ (mod\ 4)$. That is $n = 4k$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k-2)-4k) \equiv 0\ (mod\ 8) \Rightarrow m(m(6k-1)-2k) \equiv 0\ (mod\ 4) \Rightarrow m^2(6k-1)-2km \equiv 0\ (mod\ 4)$. Clearly, $2km$ is even and $6k-1$ is odd. Hence m^2 must be even. That is $m \equiv 0\ (mod\ 2)$, which is a condition (5).

Assume that $n \equiv 1\ (mod\ 4)$. That is $n = 4k+1$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+3-2)-4k-1) \equiv 0\ (mod\ 8) \Rightarrow m^2(12k+1)-4mk-m \equiv 0\ (mod\ 8) \Rightarrow 12m^2k+m^2-4mk-m \equiv 0\ (mod\ 8) \Rightarrow 4mk(3m-1)+m(m-1) \equiv 0\ (mod\ 8)$. Clearly, $4mk(3m-1) \equiv 0\ (mod\ 8)$, since $m(3m-1)$ is even. So $m(m-1) \equiv 0\ (mod\ 8)$. That implies $m \equiv 0\ (mod\ 8)$, which is a condition (1) and $m \equiv 1\ (mod\ 8)$ which is a condition (2).

Assume that $n \equiv 2\ (mod\ 4)$. That is $n = 4k+2$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+6-2)-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(m(12k+4)-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(4(m(3k+1)-k)-2) \equiv 0\ (mod\ 8) \Rightarrow m(12mk+4m-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(6mk+2m-2k-1) \equiv 0\ (mod\ 4)$. Clearly, $m \equiv 0\ (mod\ 4)$, since $6mk+2m-2k-1$ is odd, which is a condition (3).

Assume that $n \equiv 3\ (mod\ 4)$. That is $n = 4k+3$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+9-2)-4k-3) \equiv 0\ (mod\ 8) \Rightarrow m(m(12k+7)-4k-3) \equiv 0\ (mod\ 8) \Rightarrow 12mk+7m-4k-3 \equiv 0\ (mod\ 8) \Rightarrow 8mk+4mk-4k+8m-8-m+5 \equiv 0\ (mod\ 8) \Rightarrow 4k(m-1)-m+5 \equiv 0\ (mod\ 8)$. Clearly, m is odd and it satisfies when $m \equiv 5\ (mod\ 8)$ which is a condition (4).

Conversely,

1. The condition (1) is proved by using conditions 1.4 (1) and 1.7 (1) and (2).
2. The condition (2) is proved by using conditions 1.4 (5) and 1.7 (2).

3. The condition (3) is proved by using conditions 1.4 (2) and 1.7 (1).

4. The condition (4) is proved by using conditions 1.4 (4) and 1.7 (2).

5. Consider $n \equiv 0 \ (mod\ 4)$ and $m \equiv 0 \ (mod\ 2)$. The case (5) can be stated into two subcases as $n \equiv 0 \ (mod\ 4)$, $m \equiv 0 \ (mod\ 4)$ and $n \equiv 0 \ (mod\ 4)$, $m \equiv 2 \ (mod\ 4)$. For $n \equiv 0 \ (mod\ 4)$, $m \equiv 0 \ (mod\ 4)$, $K_m \boxtimes P_n$ is fork-decomposable by 1.4 (2) and 1.7 (1).

Now, let us prove the result for $n \equiv 0 \ (mod\ 4)$, $m \equiv 2 \ (mod\ 4)$. Consider $K_m \boxtimes P_4$. For $m = 2$, the graph $K_2 \boxtimes P_4$ is fork-decomposable by Lemma 2.1. $K_m \boxtimes P_4$ can be decomposed into one copy of $K_2 \boxtimes P_4$ and 10 copies of $K_{2,m-2}$. $K_{2,m-2}$ is fork-decomposable by Theorem 1.1. $K_m \boxtimes P_n$ can be decomposed into $\frac{n}{4}$ copies of $K_m \boxtimes P_4$ and $\frac{n}{4} - 1$ copes of $K_{m,m}$ which is fork-decomposable by Theorem 1.1. □

5 Strong product of complete graph and cycle

In the following theorem, we state the necessary and sufficient condition for the fork-decomposition of strong product of complete graph and cycle.

Theorem 5.1. *The graph $K_m \boxtimes C_n$ if fork-decomposable if and only if satisfies any one of the following conditions.*

1. $n \equiv 0 \ (mod\ 4)$
2. $m \equiv 0, 3 \ (mod\ 8)$
3. $m \equiv 4, 7 \ (mod\ 8)$ *and* $n \equiv 0 \ (mod\ 2)$

Proof. Total number of edges in $K_m \boxtimes P_n$ is $\frac{mn(3m-1)}{2}$.

If the graph $K_m \boxtimes C_n$ is fork-decomposable, then $\frac{mn(3m-1)}{2}$ satisfies Equation (2).This implies that $mn(3m-1) = 8k$, where k is any arbitrary positive integer. Hence our possibilities are $m = 8k$ or $n = 8k$ or $3m - 1 = 8k$ or n and m are even.

$m = 8k$, is the first part of condition (2).

Clearly, $m(3m - 1)$ is even and hence $n = 4k$ which is a condition (1).

Consider $3m - 1 = 8k$ which implies $m = \frac{8k+1}{3}$. Since m is an integer, $m \equiv 3 \ (mod\ 8)$, which is a second part of condition (2).

Let m and n are even and consider $n \equiv 0 \ (mod\ 2)$. Then $m(m - 1)$ is a multiple of 4 which implies that either $m = 4k$ or $3m - 1 = 4k$. If $m = 4k$, $m \equiv 0 \ (mod\ 4)$, which is a first part of condition (3).

Also if, $3m - 1 = 4k$ which implies $m = \frac{4k+1}{3}$. Since m is an integer, m takes the values $3, 7, 11, 15, 19, \ldots$. Hence $m \equiv 3 \ (mod\ 4)$. This congruence can be stated in two ways as $m \equiv 3 \ (mod\ 8)$ and $m \equiv 7 \ (mod\ 8)$. The condition

$m \equiv 3 \ (mod \ 8)$ is satisfied for all n which is a condition 2. Hence when $n \equiv 0 \ (mod \ 1)$. we have $m \equiv 7 \ (mod \ 8)$, which is the second part of condition (3). Conversely,

1. Consider $n \equiv 0 \ (mod \ 4)$. This condition is satisfied by using conditions 1.5 (3) and 1.8 (2).

2. Consider $m \equiv 0, 3 \ (mod \ 8)$. Consider $m \equiv 0 \ (mod \ 8)$. This condition is satisfied by using conditions 1.5 (3) and 1.8 (2). Consider $m \equiv 3 \ (mod \ 8)$. The graph $K_m \boxtimes C_n$ can be decomposed into $K_{m-3} \boxtimes C_n$, $K_3 \boxtimes C_n$ and n copies of $K_{3,3(m-3)}$. $K_3 \boxtimes C_n$ is fork-decomposable by Theorem 3.2 and $K_{3,3(m-3)}$ is fork-decomposable by Theorem 1.1.

3. Consider $m \equiv 4, 7 \ (mod \ 8)$ and $n \equiv 0 \ (mod \ 2)$. Consider $m \equiv 4 \ (mod \ 8)$. This condition is satisfied by using conditions 1.5 (1) and 1.8 (2). Consider $m \equiv 7 \ (mod \ 8)$. This condition is satisfied by using conditions 1.5 (1) and 1.8 (2). □

6 Discussion and Conclusion

In this paper, we have reviewed the literature on decomposition of graphs with special reference to the subgraph fork. Fork-decomposition of 191-edge connected graphs has already been studied in the literature. But this constant is far from best possible. Very little is known about lower bounds. In this paper, we have investigated and characterized some class of lower edge connectivity graphs for fork-decomposition. In Section 2, we have characterized the fork-decomposition of strong product of two paths. In Section 3, we have characterized the fork-decomposition of strong product of path and cycle and strong product of two cycles. In Section 4, we have characterized the fork-decomposition of strong product of complete graph and path. In Section 5, we have characterized the fork-decomposition of strong product of complete graph and cycle. A similar characterization for fork-decomposition of $G \boxtimes H$ where $G, H \in \{K_n, K_{m,n}, W_n\}$ seems to be an interesting open problem for further research.

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