Ratio Mathematica Volume 48, 2023

Amalgamated rings with m-nil clean properties

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Abstract

In this paper, we study the transfer of the notion of m-nil clean (i.e., a ring in which every element is a sum of a nilpotent and an m-potent elements) to the amalgamarted rings. We also find many sufficient and necessary conditions for the m-nil clean property to the amalgamated rings.
Keywords: Clean ring; nil-clean; m-clean; m-nil clean; m-potent.
2020 AMS subject classifications: 13A99, 13B99, 13A15, 16N20, 16N40. [1]

1 Introduction

In this paper, all our rings are assumed to be commutative with identity. Nicholson [1977] initially proposed the idea of clean rings in his research on Lifting idempotents and exchange rings. A ring is called clean if every element of the ring can be written as sum of a unit and an idempotent element. Clean rings are initially developed in Nicholson [1977] as a natural class of rings which have the exchange property. Various generalization of clean rings have been explored. Ye [2003] introduced and studied semiclean rings. Diesl [2013] initiated the notion of nil-clean ring where he replaced the unit element by a nilpotent element. S. Purkait and Kar [2020] introduced the notion of m-clean and strongly m-clean

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[1]Received on February 13, 2023. Accepted on July 15, 2023. Published on September 3, 2023. DOI: 10.23755/rm.v48i0.1133. ISSN: 1592-7415. eISSN: 2282-8214.

rings. Letting m be a positive integer greater than or equal to 2, a ring R is said to be m-clean if each element of R can be written as a sum of a unit and an m-potent element (i.e., an element r of a ring R is said to be m-potent if $r^m = r$). Following Diesl [2013], an element of a ring is called nil-clean if it is a sum of an idempotent and a nilpotent, and a ring is called nil-clean if every element is nil-clean. R. Barati and Abyzov [2022] gave a generalization to nil-clean called m-nil clean. An element r of a ring R is said to be m-nil clean if $r = n + f$, where f is m-potent in R and n is nilpotent in R. A ring R is called m-nil clean if each of its elements is m-nil clean. For a fixed natural number $m > 1$, the class of m-nil clean rings is a big subclass of the periodic rings which contains the class of nil-clean rings. A ring R is called periodic if for each $r \in R$ there exist distinct positive integers m and n such that $r^m = r^n$. A ring R is called potent if each element of R is potent (i.e., $r \in R$ is potent if $r^n = r$ for some $n \in \mathbb{R}$). Clearly, potent rings are periodic. For a fixed natural number $m > 1$, a ring R is called m-potent if $r^n = r$ for every $r \in R$. A ring R is UU-ring if every unit u in R is a unipotent, i.e., u can be expressed as $u = 1 + n$, where n is a nilpotent element of ring R.

Let R and S be rings, J be an ideal of S and $\phi : R \to S$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times S$:

$$R \bowtie^{\phi} J := \{(r, \phi(r) + j) \mid r \in R, j \in J\}$$

called *the amalgamation of R with S along J with respect to ϕ* (introduced and studied by M. D'Anna and Fontana [2009], M. D'Anna and Fontana [2010]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in (D'Anna and Fontana [2007a], D'Anna and Fontana [2007b])).

In this paper, we investigate the structure of the rings of the form $R \bowtie^{\phi} J$, with a particular attention to the m-nil clean ring property. More precisely, we give a necessary and sufficient condition for the amalgamated rings to be m-nil clean. For the convenience of the reader, we denote $U(R)$, $M(R)$, $Idem(R)$, $Nil(R)$ and $J(R)$, the set of all unit elements of R, the set of all m-potent elements of R, the set of all idempotent elements of R, the set of all nilpotent elements of R and the Jacobson radical of R, respectively.

2 m-nil clean Ring

The aim of this section is to investigate general structure of m-nil clean rings which will be used in the sequel.

Definition 2.1. *R. Barati and Abyzov [2022] Let R be a ring. An element $r \in R$ is called m-nil clean if there exist a nilpotent $n \in R$ and an m-potent element $f \in R$ such that $r = n + f$. A ring R is called m-nil clean if every element of R is m-nil clean.*

Example 2.1. *The class of m-nil clean rings contains many familiar examples.*

1. *Every strongly m-nil clean ring is an m-potent ring.*
2. *Any queotient of an m-nil clean ring is m-nil clean.*
3. *Any finite product of m-nil clean ring is m-nil clean. But $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$ is an infinite product of m-nil clean rings, which is not m-nil clean. The element $(0, 2, 2, \cdots \in R)$ cannot be written as the sum of an m-potent and a nil potent element.*
4. *If R is m-nil clean, then so is the ring $R[[x]]/(x^n), n \in \mathbb{N}$*

Lemma 2.1. *If $r \in R$ be an m-potent element, then r^{m-1} is an idempotent element in R.*

Proof. Let $r \in R$ is an m-potent element. Then $(r^{m-1})^2 = r^{2(m-1)} = r^{2m-2} = r.r^{m-2} = r^{m-1}$. □

Proposition 2.1. *Let R be a commutative ring. Then*

(1) If $r \in R$ is a m-nil clean element, then r^n is a nil-clean element.

(2) If $r \in R$ is a m-nil clean element, then $r^n - r^{2n}$ is a nilpotent for some n.

Proof. (1) Assume that $r \in R$ is an m-nil clean element. Then, we have $r = f + n$ with $n \in Nil(R)$ and $f \in M(R)$. This implies $(r - n)^{m-1} = f^{m-1}$ is an idempotent by Lemma 2.1 . By applying Binomial theorem we get our desired result.
(2) Suppose $r \in R$ is a m-nil clean element. Then by (1), r^n is a nil-clean element. Then by [Y. Hirano and A.Yaqub, 1988, Theorem 3], $r^n - r^{2n}$ is nilpotent. □

Lemma 2.2. *Let R be a ring and $m > 1$ be an integer. Then R is strongly m-nil clean if and only if $r^m - r$ is nilpotent for all $r \in R$.*

Proof. It follows from [R. Barati and Abyzov, 2022, Theorem 1.10]. □

Proposition 2.2. *A ring R is m-nil clean if and only if $R/Nil(R)$ is an m-potent ring.*

Proof. Let $r + Nil(R) \in R/Nil(R)$. Since R is m-nil clean, r can be written as $r = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. So, $r + Nil(R) = n + f + Nil(R) = f + Nil(R)$ and it is easy to see that $(f + Nil(R))^m = f + Nil(R)$. Therefore, $R/Nil(R)$ is an m-potent ring.
Conversely, assume that $R/Nil(R)$ is an m-potent ring. Let $r \in R$, then $r + Nil(R)$ is m-potent, i.e., $(r + Nil(R))^m = r + Nil(R)$. So $r^m - r \in Nil(R)$ and then, by Lemma 2.2, R is m-nil clean. □

The following proposition provides a characterizations of m-nil clean rings.

Proposition 2.3. *R is an m-nil clean ring if and only if for each $r \in R$, we have $r = f + u$, where $f \in M(R)$ and $u \in UU(R)$*

Proof. ($\Rightarrow$) Assume that R is an m-nil clean ring. Let $r \in R$. Then we have $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. This implies $r = u + f$ such that $u = 1 + n \in UU(R)$ and $f \in M(R)$.
($\Leftarrow$) Let us assume that each element $r \in R$, we have $r = f + u$, where $u \in UU(R)$ and $f \in M(R)$. Let $r \in R$, then $r + 1 = u + f$ with $u \in UU(R)$ and $f \in M(R)$. This implies that $r + 1 = 1 + n + f$ such that $n \in Nil(R)$. Therefore, $r = n + f$. Hence, R is m-nil clean. □

Proposition 2.4. *Every m-nil clean ring is an m-clean ring.*

Proof. Assume that R is an m-nil clean ring. Let $r \in R$, then $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. This implies $r = 1 + n + f$. Therefore, $r = u + f$, with $1 + n = u \in U(R)$. Hence, R is m-clean. □

The following example illustrate the converse of the Proposition 2.4 is not true.

Example 2.2. *Every infinite field is an m-clean ring, but not a m-nil clean.*

Proposition 2.5. *If R is m-clean and UU-ring, then R is an m-nil clean ring.*

Proof. Assume that R is both m-clean and UU-ring. Let $r \in R$, then $r - 1 = u + f$, where $u \in U(R)$ and $f \in M(R)$. Since R is UU-ring, $u = 1 + n$ for some $n \in Nil(R)$. Therefore, $r = n + f$. Hence, R is m-nil clean. □

Definition 2.2. *S. Purkait and Kar [2020] Let I be an ideal of a ring R. We say that m-potents lifted modulo I if for any $r \in R$ with $r - r^m \in I$ implies that there exists an m-potent f in R such that $f - r \in I$*

Proposition 2.6. *Let R be a UU-ring and I be an ideal of R such that $I \subset J(R)$. If R/I is m-nil clean and if m-potents in R can be lifted modulo I, then R is m-nil clean.*

Proof. Let us use $\overline{r}$ to denote $r + I$ in R/I. Let $r \in R$, then we can write $\overline{r} = \overline{n} + \overline{f}$, where $\overline{n} \in Nil(R/I)$ and $\overline{f} \in M(R/I)$. By assumption, we may assume $f^m = f$. By the fact that $\overline{1 + r - f} = \overline{1 + n}$ is a unit in R/I and $I \subseteq J(R)$. Now, it is easy to see that $1 + r - f$ is a unit in R. It follows immediately that $r = n + f$. □

Proposition 2.7. *Let R be a ring. Then the polynomial ring $R[x]$ is not an m-nil clean ring.*

Proof. Assume that $x = n + f$, where $f^m = f$ and $f \in R[x]$ and n is nilpotent in $R[x]$. By the commutativity of R, f must be in R. Thus $-f + x$ is nilpotent in $R[x]$. It follows immediately that 1 is a nilpotent element in R, which is impossible. □

Lemma 2.3. *Let R be an m-nil clean ring. Then $J(R)$ is nil.*

Proof. Let $r \in J(R)$. Then $r = f + n$, where $f \in M(R)$ and $n \in Nil(R)$. Suppose $n^k = 0$ for some k. Then $(r - f)^{mk+m-k} = n^{mk+m-k} = 0$; hence, $f^{mk+m-k} \in J(R)$. clearly $f^{mk+m-k} = (f^m)^k.f^{m-k} = f^k.f^{m-k} = f^m = f$. It follows that $f \in J(R)$, and so $f(f^{m-1} - 1) = 0$. This implies that $f = 0$; hence, $r = n \in Nil(R)$. Therefore, $J(R)$ is nil, as asserted. □

Theorem 2.1. *Let R be a ring such that $Idem(R) = \{0, 1\}$. If R is an m-nil clean ring, then each element of R is m-clean or sum of two units.*

Proof. Let $r \in R$. Since R is an $m$nil clean ring, $r - 1$ is an m-nil clean element, i.e., $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. Then f^{m-1} is idempotent belongs to $Idem(R) = \{0, 1\}$, i.e., $f^{m-1} = 0$ or $f^{m-1} = 1$. Now if $f^{m-1} = 0$, then $r - 1 = n$. Since n is nilpotent then $r = n + 1$ is unit. If $f^{m-1} = 1$, then f is unit and $r = f + n + 1$, so r is the sum of two units. □

3 Amalgamated rings with m-nil clean properties

In this section, we investigate the m-nil clean properties of the amalgamated ring $R \bowtie^{\phi} J$. We start with an example to illustrate the definition.

Example 3.1. *Let $R = \mathbb{Z}_2$ and $S = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ be rings and $J = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ an ideal of S and $\phi : R \to S$ defined by $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, where $r \in \mathbb{Z}_2$. Then R, S and $\phi(R) + J$ are m-clean rings. The amalgamated ring*

$R \bowtie^{\phi} J = \left\{ \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \right\}$ *is also an m-nil clean ring.*

In the following results, we prove that R and $\phi(R) + J$ are m-nil clean whenever the amalgamated ring $R \bowtie^{\phi} J$ is m-nil clean. We recall that a ring R is said to be uniquely m-nil clean if every element of the ring can be uniquely written as sum of nilpotent and m-potent element.

Lemma 3.1. *Let R be a ring and $m > 1$ be an integer. Then R is m-nil clean if and only if $r^m - r$ is nilpotent for all $r \in R$.*

Proof. It follows from [R. Barati and Abyzov, 2022, Theorem 1.10] □

Theorem 3.1. *let $\phi : R \to S$ be a ring homomorphism and let J be an ideal of S. Then the following conditions are equivalent :*

1. *$R \bowtie^{\phi} J$ is m-nil clean.*
2. *R and $\phi(R) + J$ are m-nil clean.*

Proof. $(1) \Rightarrow (2)$: If $R \bowtie^{\phi} J$ is m-nil clean, then R and $\phi(R) + J$ are homomorphic images of $R \bowtie^{\phi} J$ (see [M. D'Anna and Fontana, 2009, Proposition 5.1],). Hence R and and $\phi(R) + J$ are m-nil clean.
$(2) \Rightarrow (1)$: Let $(r, \phi(r)+j) \in R \bowtie^{\phi} J$. Then R and and $\phi(R)+J$ are m-nil clean implies $r^m - r$ and $(r, \phi(r)+j)^m - (r, \phi(r)+j)$ are nilpotent elements in R and and $\phi(R)+J$, respectively. Therefore, it is easy to see that $(r, \phi(r)+j)^m - (r, \phi(r)+j)$ is nilpotent in $R \bowtie^{\phi} J$. Hence $R \bowtie^{\phi} J$ is m-nil clean. □

Example 3.2. *Let $R = \mathbb{Z}_6, S = \mathbb{Z}_3 \times \mathbb{Z}_3, J = 0 \times \mathbb{Z}_3$ and $\phi : R \to S$ defined by $\phi(0) = \phi(3) = (0,0), \phi(1) = \phi(4) = (1,1)$ and $\phi(2) = \phi(5) = (2,2)$. It is easy to see that R and $\phi(R)+j = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ are m-nil clean rings. Then by Theorem 3.1, the amalgamated ring $R \bowtie^{\phi} J = \{(0,(0,0)),(0,(0,1)),(0,(0,2)),(1,(1,0)),(1,(1,1)),(1(1,2)),(2,(2,0)),(2,(2,1)),(2,(2,2))(3,(0,0)),(3,(0,1)),(3,(0,2)),(4,(1,0)),(4,(1,1)),(4,(1,2))(5,(2,0)),(5,(2,1)),(5,(2,2))\}$ is m-nil clean.*

Corolary 3.1. *Let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S. Then the following conditions are equivalent:*

1. *$R \bowtie^{\phi} J$ is m-nil clean.*
2. *Any proper homomorphic image of $R \bowtie^{\phi} J$ is m-nil clean.*

Proof. (1) $\Rightarrow$ (2) Assume that $R \bowtie^{\phi} J$ is m-nil clean. It is well known that any homomorphic image of a m-nil clean ring is m-nil clean (due to the fact that any homomorphic image of m-potent is m-potent and homomorphic image of nilpotent is nilpotent).
(2) $\Rightarrow$ (1) Assume that any homomorphic image of $R \bowtie^{\phi} J$ is m-nil clean. Since R and $\phi(R) + J$ are proper homomorphic image of $R \bowtie^{\phi} J$, so R and $\phi(R) + J$ are m-nil clean by Theorem 3.1, as desired. □

Corolary 3.2. *Let $\phi : R \to S$ be a ring homomorphism and R is m-nil clean and J be an ideal of S with $J \subset Nil(S)$, then $R \bowtie^{\phi} J$ is m-nil clean.*

Proof. By Theorem 3.1, it is enough to prove that $\phi(R) + J$ is m-nil clean. Let $(r, \phi(r) + j) \in R \bowtie^{\phi} J$. Since R is m-nil clean, $r = f + n$, where f is an m-potent and n is a nilpotent. Therefore, $\phi(r) + j = \phi(f) + \phi(n) + j$. Since $J \subset Nil(S), j \in Nil(S)$ and $\phi(n) \in Nil(R)$ implies that $\phi(n) + j \in Nil(\phi(R) + J)$. Therefore, $\phi(r) + j$ is a sum of m-potent $\phi(f)$ and nilpotent $\phi(n) + j$ in $\phi(R) + J$, as desired. □

Corolary 3.3. *let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S such that $J \cap M(B) = 0$ and $M(R \bowtie^{\phi} J) = \{(f, \phi(f)) | f \in M(R)\}$. Then the following conditions are equivalent:*

1. *$R \bowtie^{\phi} J$ is m-nil clean.*
2. *R is m-nil clean and $J \subset Nil(S)$*

Proof. By Theorem 3.1, R is m-nil clean and now it remains to prove that $J \subset Nil(S)$.
Let $j \in J$, without loss of generality, we may assume that $j \neq 0$. Hence $(0, j)$ is an m-nil clean element of $R \bowtie^{f} J$ and so by [M. Chhiti and Tamekkante, 2015, Lemma 2.10], $(0, j) = (f, \phi(f)) + (n, \phi(n) + k)$, where $f \in M(R), n \in Nil(R)$ and $k \in Nil(S) \cap J$. Since $f + n = 0$, $f = -n \in M(R) \cap Nil(R) = \{0\}$ and so $j = \phi(f) + \phi(n) + k = k$. Therefore $J \subset Nil(S)$.
Conversely, assume that R is m-nil clean and $J \subset Nil(S)$. Then, by Corollary 3.2, $R \bowtie^{\phi} J$ is m-nil clean. □

Theorem 3.2. *let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S and $J \subseteq Nil(S)$. Then we have the following statements:*

1. *Assume that ϕ is one-one. If $\phi(R) + J$ is m-nil clean, then $R \bowtie^{\phi} J$ is m-nil clean.*
2. *Assume that J is nilpotent. Then $R \bowtie^{\phi} J$ is m-nil clean if and only if R is m-nil clean.*

Proof.
(1) As $J \subseteq Nil(S)$ and ϕ is one-one, $\phi^{-1}(J) \subseteq Nil(R)$. On the other hand $\frac{R \bowtie^\phi J}{\phi^{-1}(J)\{0\}} \cong \phi(R) + J$. The remaining proof is similar to (1).
(2) The forward direction is shown in Theorem 3.1.
Conversely, let $(r, \phi(r) + j) \in R \bowtie^\phi J$. By the fact that $\frac{R \bowtie^\phi J}{\{0\} \times J} \cong R$, we get $\overline{(r, \phi(r) + j)} \in \frac{R \bowtie^\phi J}{\{0\} \times J}$ is an m-nil clean element, as R is m-nil clean. □

Theorem 3.3. *Let $\phi : R \to S$ be a ring homomorphism and let J be an ideal of S. Set $\overline{R} = R/Nil(R), \overline{S} = S/Nil(S), \pi : S \to \overline{S}$ the canonical projection and $\overline{J} = \pi(J)$ and consider the ring homomorphism $\overline{\phi} : \overline{R} \to \overline{S}$ such that $\overline{\phi}(\overline{r}) = \overline{\phi(x)}$. Then, $R \bowtie^\phi J$ is m-nil clean if and only if $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is an m-potent ring.*

Proof. It is easy to see that $\overline{\phi}$ is well defined and it is a ring homomorphism. The map $\xi : R \bowtie^\phi J/Nil(R \bowtie^\phi J) \to \overline{R} \bowtie^{\overline{\phi}} \overline{J}$ defined by $\xi(\overline{r, f(r) + j}) = (\overline{r}, \overline{\phi}(\overline{r}) + \overline{j})$ is an isomorphism.
Assume that $R \bowtie^\phi J$. Then by Proposition 2.2, $R \bowtie^\phi J/Nil(R \bowtie^\phi J)$ is m-potent and so $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is m-potent.
Conversely, assume that $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is m-potent. Hence $R \bowtie^\phi J/Nil(R \bowtie^\phi J)$ is m-potent and so $R \bowtie^\phi J$ is m-nil clean by Proposition 2.2. □

Example 3.3. *Let T be a ring, J be an ideal of T, and let D be a subring of T such that $J \cap D = (0)$. If $J \subset Nil(T)$, then the ring $D + J$ is m-nil clean if and only if D is m-nil clean.*

Proof. By [M. D'Anna and Fontana, 2009, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^i J$, where $i : D \hookrightarrow J$ is the natural embedding. Thus, by Theorem 3.2, $D + J$ is m-nil clean if and only if D is m-nil clean. □

4 Conclusions

In this paper, we provide the necessary and sufficient condition for amalgamated ring to be a m-nil clean ring. This study will further help in studying the m-nil clean properties in other classical constructions in the ring theory. In future, there is a scope to study the generalization of amalgamated ring namely bi-amalgamated ring with m-nil clean properties.

Acknowledgments

The first author is supported by CSIR-UGC NET JRF (F.No. 16-9 (June 2018)/2019(NET/CSIR) dated 16th April 2019). The second author is supported by DST FIST (Letter No: SR/FST/MSI-115/2016 dated 10th November 2017)

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